Popular time series parametric models

(Hamilton: Chapter 3)

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Stationary ARMA processes

- White noise process
- Moving average process
- Autoregressive process
- Autoregressive moving average process

White noise

$$y_t = \epsilon_t \quad \epsilon_t \sim IID(\mu, \sigma^2)$$

- It is weakly stationary.
- The mean and variance do not depend on time
- If the distribution of the epsilons is normal, then it is called a Gaussian white noise.
- A Gaussian white noise is strictly stationary because a multivariate normal distribution is completely define with the mean and variance-covariance matrix.
- The epsilons are uncorrelated $\gamma_h = cov(\epsilon_t, \epsilon_{t+h}) = 0$ for $h \neq 0$
- This process is completely unpredictable.

Simulating and estimating White noise in R

Simulate samples of size 200 of the following processes:

$$y_t = \epsilon_t \quad \epsilon_t \sim IIDN(\mu, \sigma^2)$$
$$y_t = \epsilon_t \quad \epsilon_t \sim IID\chi^2_9(\mu, \sigma^2)$$
$$y_t = \epsilon_t \quad \epsilon_t \sim IID(\mu, \sigma^2)$$

```
y1=rnorm(200, 20, sd=4.3)
y2=rchisq(200,df=9 )
plot(y1)
plot(y2)
```

Linear time series

A times series y_t is said to be linear if it can be written as an MA(∞) model:

$$y_t = \mu + \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i}$$

- μ is the mean of y_t
- $\psi_0 = 1$
- $\{\epsilon_t\}$ is a white noise. It represents the new information in time t (innovation, shock)

Linear time series

- The dynamics of linear time series are govern by the parameters ψ_i
- If y_t is weakly stationary then we can obtain its mean, variance and covariance.
- ARMA processes are the most popular of linear time series
- ARMA processes are stationary.

Stationary MA processes

- MA(1) – file: proof1.pdf

Simulating and estimating MA in R

Given the MA(1) proccess

$$y_t = 20 + 0.7\epsilon_{t-1} + \epsilon_t \quad \epsilon IIDN(0, 1)$$

What is its mean? and the intercept?

Simulate the proccess using *arima.sim* and estimate it using function *arima*. What is the mean? and the intercept?

Simulating and estimating MA in R

Given the MA(1) proccess

$$y_t = 20 + 0.7\epsilon_{t-1} + \epsilon_t \quad \epsilon IIDN(0,1)$$

The mean is 20 and the intercept is 20.

```
> y= arima.sim(1000, model=list(ma= 0.7)) + 20
> mean(y)
[1] 19.92846
> arima(y, order=c(0,0,1))
Coefficients:
         ma1  intercept
      0.6651    19.9278
s.e.  0.0249     0.0511

sigma^2 estimated as 0.9434:  log likelihood = -1390.09,  a
```

The qth order Moving Average Process (MA(q))

Y_t is constructed from a weighted sum of the q most recent realizations of the error term.

Definition (MA(q)):

An MA(q) process Y_t is defined as

$$Y_t = \mu + \epsilon_t + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} = \mu + \theta(L)\epsilon_t,$$

where ϵ_t is a zero mean $i.i.d.$ process and $(\mu, \theta_1, ..., \theta_q) \in \mathbb{R}^{q+1}$, is a vector of parameters.

The qth order Moving Average Process (MA(q))

- The mean of Y_t:

$$\mathrm{E}(Y_t) = \mu.$$

- The variance of Y_t:

$$\gamma_0 \equiv \mathrm{var}(Y_t) = (1 + \theta_1^2 + \ldots + \theta_q^2)\mathrm{var}(\epsilon_t).$$

- The autocovariance function of Y_t for $j = 1, 2, \ldots, q$:

$$\gamma_j \equiv \mathrm{cov}(Y_t, Y_{t-j}) = (\theta_j + \theta_{j+1}\theta_1 + \theta_{j+2}\theta_2 + \ldots + \theta_q\theta_{q-j})\,\mathrm{var}(\epsilon_t).$$

- Notice that for $j > q$, $\mathrm{cov}(Y_t, Y_{t-j}) = 0$.

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The Infinite-Order Moving Average Process (MA(∞)):

According to the definition above the MA(∞) process can be written as

$$Y_t = \mu + \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}, \quad \psi_0 = 1.$$

- Question: Under which circumstances will the MA(∞) process be covariance stationary and ergodic?
- Answer: When $\sum_{j=0}^{\infty} |\psi_j| < \infty$ (for a proof, see notes from class).
- Note: Covariance stationarity only requires that the sequence $\{\psi_j^2\}_{j=0}^{\infty}$ is square summable (absolute summability implies square summability).

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First order autorregressive process (AR(1))

Definition:

The first order (Gaussian) autoregressive process (AR(1)) is defined as

$$\begin{aligned} Y_t &= c + \phi Y_{t-1} + \epsilon_t, \\ \epsilon_t &\sim N(0, \sigma^2). \end{aligned}$$

Proposition:

Let Y_t be a first order autoregressive process. If $|\phi| < 1$, Y_t will be covariance stationary and ergodic in the mean.

Stationary AR processes

- AR(1) – file: proof1.pdf

Simulating and estimating AR in R

Given the AR(1) proccess

$$y_t = 20 + 0.7y_{t-1} + \epsilon_t \quad \epsilon_t IIDN(0, 1)$$

What is its mean? and the intercept?

Simulate a sample of size 1000 of this proccess using *arima.sim* and estimate it using function *arima*.

What is the mean? and the intercept?

Simulating and estimating AR in R

$$y_t = 20 + 0.7 y_{t-1} + \epsilon_t \quad \epsilon IIDN(0,1)$$

The mean is 20/(0.3)=66.67, the intercept is 20

```
> set.seed(20)
> y= arima.sim (1000, model=list(ar=0.7))+66.67
> mean(y)
[1] 66.6577
> ar1= arima(y, order=c(1,0,0)) #note include.mean=T
> ar1

Coefficients:
         ar1  intercept
      0.6820    66.6614
s.e.  0.0231     0.1007

sigma^2 estimated as 1.03:  log likelihood = -1433.81,  aic = 2873.62
```

Simulating and estimating AR in R

$$y_t = 20 + 0.7 y_{t-1} + \epsilon_t \quad \epsilon IIDN(0,1)$$

The mean is 20/(0.3)=66.67, the intercept is 20

```
> ar2=ar(y, method="mle")
> ar2$x.mean
[1] 66.6577
> ar$coef[2]*(1-ar$coef[1])
[1] 21.20
```

Simulating and estimating AR in R

The *arima* function says that the model is:

$$y_t = 66.66 + 0.68 y_{t-1} + \epsilon_t$$

However where it is written intercept, in reality it should say mean. The intercept $c = \mu(1 - \phi)$. So, in reality the model is:

$$y_t = 21.20 + 0.68 y_{t-1} + \epsilon_T$$

Simulating and estimating AR in R

```
> y2=arima.sim(1000, model=list(ar=0.7),
innov =rnorm(1000, 20, 1))
> mean(y2)
[1] 66.49881
> 20/(1-0.7)
[1] 66.66667
> ar3=arima(y2, order=c(1,0,0))
> ar3

Coefficients:
         ar1  intercept
      0.9961    63.5754
s.e.  0.0054     9.1600

sigma^2 estimated as 1.616:  log likelihood = -1661.5,  aic = 33

> 63.5754*(1-0.9961)
[1] 0.2479441
```

Simulating and estimating AR in R

The *arima* function says that the model is:

$$y_t = 0.25 + 0.996 y_{t-1} + \epsilon_t$$

which is a bad prediction

Simulating and estimating AR in R

```
> ar4=ar(y2)
> ar4

Coefficients:
     1
0.6992

Order selected 1  sigma^2 estimated as  3.187
> ar4$x.mean
[1] 66.668
> ar4$x.mean*(1-ar4$ar)
[1] 20.05435
```

$$y_t = 20.05 + 0.699 y_{t-1} + \epsilon_t$$

Simulating and estimating AR in R

```
> ar5=ar(y2, method="mle")
> ar5

Call:
ar(x = y2, method = "mle")

Coefficients:
     1       2       3       4
0.9163  0.0196  0.0381  0.0240

Order selected 4  sigma^2 estimated as  1.49
```

First order autorregressive process (AR(1))

Proof:

Rewrite the AR(1) process as

$$(1-\phi L)\,Y_t = c + \epsilon_t, \quad |\phi| < 1$$

and notice that provided $|\phi| < 1$ the inverse of $(1-\phi L)$ exists and is given as

$$(1-\phi L)^{-1} = \lim_{q\to\infty} \sum_{j=0}^{q} \phi^j L^j, \quad |\phi| < 1.$$

Consequently,

$$Y_t = \lim_{q\to\infty} \sum_{j=0}^{q} \phi^j c + \lim_{q\to\infty} \sum_{j=0}^{q} \phi^j L^j \epsilon_t, \quad |\phi| < 1. \qquad (1)$$

$$Y_t = \frac{c}{1-\phi} + \sum_{j=0}^{\infty} \phi^j \epsilon_{t-j}, \quad |\phi| < 1.$$

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First order autorregressive process (AR(1))

Proof (continued):
Hence Y_t also have an $MA(\infty)$ representation whenever $|\phi| < 1$ with $\psi_j = \phi^j$ for $j = 0, 1, 2, \ldots$.

Recall that a sufficient condition for the $MA(\infty)$ process to be covariance stationary and ergodic in mean is that $\sum_{j=0}^{\infty} |\psi_j| = \sum_{j=0}^{\infty} |\phi|^j = \frac{1}{1-|\phi|} < \infty$ which holds whenever $|\phi| < 1$.

Moments of the $AR(1)$ process ($|\phi| < 1$):

Mean (unconditional):

$$\mathsf{E}(Y_t) = \mathsf{E}\left(\frac{c}{1-\phi}\right) + \mathsf{E}\left(\sum_{j=0}^{\infty} \phi^j \epsilon_{t-j}\right) = \frac{c}{1-\phi}$$

Variance (unconditional):

$$\begin{aligned}
\gamma_0 &= \lim_{q\to\infty} \mathsf{E}\left((\epsilon_t + \phi\epsilon_{t-1} + \ldots + \phi^q \epsilon_{t-q})^2\right) \\
&= \lim_{q\to\infty} \sum_{j=0}^{q} \phi^{2j} \sigma^2 \\
&= \frac{\sigma^2}{1-\phi^2}
\end{aligned}$$

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Moments of the $AR(1)$ process ($|\phi| < 1$):

Autocovariance (unconditional):

$$\begin{aligned}
\gamma_j &= \lim_{q\to\infty} \mathsf{E}\left((\epsilon_t + \phi\epsilon_{t-1} + \ldots + \phi^q\epsilon_{t-q})(\epsilon_{t-j} + \phi\epsilon_{t-1-j} + \ldots + \phi^q\epsilon_{t-q}\right. \\
&= \lim_{q\to\infty} (\phi^j + \phi^{j+2} + \phi^{j+4} + \ldots + \phi^{2q-j})\sigma^2 \\
&= \lim_{q\to\infty} \phi^j(1 + \phi^2 + \phi^4 + \ldots + \phi^{2(q-j)})\sigma^2 \\
&= \frac{\phi^j\sigma^2}{1-\phi^2}
\end{aligned}$$

Autocorrelation (unconditional)

$$\rho_j = \frac{\gamma_j}{\gamma_0} = \phi^j$$

The $AR(p)$ process and its moments:

Definition:

The (Gaussian) pth order autoregressive process Y_t is defined as

$$\begin{aligned} Y_t &= c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \epsilon_t \quad (2) \\ \phi(L) Y_t &= c + \epsilon_t \end{aligned}$$

for $\epsilon_t \sim N(0, \sigma^2)$.

The $AR(p)$ process and its moments:

Important:

- The $AR(p)$ process has a state space representation given as

$$\begin{bmatrix} Y_t \\ Y_{t-1} \\ \dots \\ Y_{t-p+2} \\ Y_{t-p+1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_{p-1} & \phi_p \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ Y_{t-2} \\ \dots \\ Y_{t-p+1} \\ Y_{t-p} \end{bmatrix} + \begin{bmatrix} \epsilon_t \\ 0 \\ \dots \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{Y}_t = \mathbf{F}\mathbf{Y}_{t-1} + \varepsilon_t$$

The $AR(p)$ process and its moments:

- From the state space representation we can find (as the solution to a "first" order difference equation)

$$\mathbf{Y}_{t+j} = \mathbf{F}^{j+1}\mathbf{Y}_{t-1} + \sum_{i=0}^{j} \mathbf{F}^{j-i}\varepsilon_{t+i}$$

- If all the eigenvalues of $\mathbf{F}$ are less than one in modulus the "system" is stable (how is this related to covariance stationarity and ergodicity?)

The $AR(p)$ process and its moments:

The process can be also be written as:

$$\phi(L)y_t = \epsilon_t$$

Provided that the roots of the lag polynomial $\phi(z)$ lies outside the unit-circle, we can write $\psi(L) = \phi(L)^{-1}$ and obtain the following covariance stationary $MA(\infty)$ representation

$$\begin{aligned} Y_t &= \psi(L)c + \psi(L)\epsilon_t \\ &= \psi(1)c + \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}. \end{aligned} \tag{3}$$

For a proof of (3), see notes from class.

The $AR(p)$ process and its moments:

Mean (unconditional):

$$\mathsf{E}(Y_t) = \psi(1)c = \frac{c}{1 - \phi_1 - ... - \phi_p} = \mu.$$

Variance and Autocovariances (unconditional):

The AR(p) process can be written as:

$$Y_t - \mu = \phi_1 (Y_{t-1} - \mu) + \phi_2 (Y_{t-2} - \mu) + ... + \phi_p (Y_{t-p} - \mu) + \epsilon_t,$$

and

$$\begin{aligned}
\gamma_0 &= \mathsf{E}(Y_t - \mu)^2 \\
&= \mathsf{E}(\phi_1 (Y_t - \mu)(Y_{t-1} - \mu) + \phi_2 (Y_t - \mu)(Y_{t-2} - \mu) + \\
&\quad ... + \phi_p (Y_t - \mu)(Y_{t-p} - \mu) + (Y_t - \mu)\epsilon_t) \\
&= \phi_1\gamma_1 + \phi_2\gamma_2 + ... + \phi_p\gamma_p + \sigma^2.
\end{aligned}$$

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The $AR(p)$ process and its moments:

In general

$$
\begin{aligned}
\gamma_j &= \mathsf{E}(Y_t - \mu)(Y_{t-j} - \mu) \\
&= \mathsf{E}(\phi_1 (Y_{t-1} - \mu) (Y_{t-j} - \mu) + \phi_2 (Y_{t-2} - \mu) (Y_{t-j} - \mu) + \\
&\quad ... + \phi_p (Y_{t-p} - \mu) (Y_{t-j} - \mu) + (Y_{t-j} - \mu) \epsilon_t) \\
&= \phi_1 \gamma_{j-1} + \phi_2 \gamma_{j-2} + ... + \phi_p \gamma_{j-p}.
\end{aligned}
$$

The $AR(p)$ process and its moments:

Comments:

- In order to find $\gamma_0, ..., \gamma_p$, as functions of $\phi_1, ..., \phi_p$ and σ^2 we must solve a system of $(p+1)$ equations.
- Numerically the $(p, 1)$ vector $(\gamma_0, \gamma_1, ..., \gamma_{p-1})'$ can be found as the first p elements of the first column of the (p^2, p^2) matrix
$$\sigma^2 \left(\mathbf{I}_{p^2} - (\mathbf{F} \otimes \mathbf{F})\right)^{-1}$$

The $AR(p)$ process and its moments

Autocorrelation (unconditional):

$$\rho_j = \frac{\gamma_j}{\gamma_0}, \quad j = 0, ..., p$$

$$\rho_j = \begin{cases} 1 & j = 0 \\ \phi_1\rho_{j-1} + \phi_2\rho_{j-2} + ... + \phi_p\rho_{j-p} & j = 1, ..., p \end{cases} \tag{4}$$

Comments:

- The system of equations given by (4) are the so-called Yule-Walker equations.
- The autocovariances and autocorrelations follow the same pth order difference equation as does the process itself.

Stationary ARMA processes

- ARMA(1,1) – file: proof1.pdf

The Autorregressive Mean Average Process (ARMA(p, q))

Definition:

The process $Y_t, t = 1, 2, ..., T$ is said to have a Gaussian ARMA(p,q) representation if

$$\begin{aligned}\phi(L)\,Y_t &= c + \theta(L)\epsilon_t, \quad \epsilon_t \sim N(0, \sigma^2),\\ \phi(L) &= 1 - \phi_1 L - \phi_2 L^2 - ... - \phi_p L^p,\\ \theta(L) &= 1 + \theta_1 L + \theta_2 L^2 + ... + \theta_q L^q.\end{aligned}$$

ARMA(p,q)

Infinite MA representation of the ARMA(p,q) process:

If the roots of $\phi(z)$ all lie outside the unit circle, $\phi(L)^{-1}$ exists and consequently we can write

$$\begin{aligned} Y_t &= \phi(L)^{-1}c + \phi(L)^{-1}\theta(L)\epsilon_t \\ &= \phi(L)^{-1}c + \psi(L)\epsilon_t, \end{aligned}$$

where $\sum_{j=0}^{\infty} |\psi_j| < \infty$, and $\psi_0 = 1$.

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ARMA(p,q)

Important result:

Stationarity of an ARMA(p,q) process depends only on the autoregressive parameters, i.e., $\phi(L)$, and NOT on the moving average parameters, i.e., $\theta(L)$.

ARMA(p,q)

Autocovariance function:

Writing the ARMA(p,q) process in deviations from the mean gives

$$\begin{aligned} Y_t - \mu &= \phi_1(Y_{t-1} - \mu) + ... + \phi_p(Y_{t-p} - \mu) + \\ &\quad \epsilon_t + \theta_1\epsilon_{t-1} + ... + \theta_q\epsilon_{t-q}` \end{aligned}$$

By multiplying both sides by $Y_{t-j} - \mu$ for $j > q$ and taking expectations yields

$$\gamma_j = \phi_1\gamma_{j-1} + \phi_2\gamma_{j-2} + ... + \phi_p\gamma_{j-p}$$

since E($Y_{t-j} - \mu)\epsilon_{t-q} = 0$ for $j > q$. The autocovariance function for the ARMA(p,q) process will be more complex than for the AR(p) process for lags 1 through q. This is due to the correlation between Y_{t-j} and ϵ_{t-q} for $j \leq q$ (For an ARMA(1,1) case, see notes from class).

Invertability of the MA(q) process

- We have seen how an AR(p) process can be given an infinite order MA process representation given the roots of the lag polynomial lie outside the unit circle. In this case we could "invert" the lag-polynomial.
- Question: Under which conditions can we "invert" an MA(q) process to obtain an infinite order AR representation?

Invertability of the MA(q) process

Example:

Consider the MA(1) process

$$Y_t - \mu = (1 + \theta L)\epsilon_t, \quad \epsilon_t \sim IID(0, \sigma^2)$$

and assume that $|\theta| < 1$, such that $(1 + \theta L)^{-1}$ exists. Notice, that

$$\begin{aligned} (1 + \theta L)^{-1} &= (1 - (-\theta)L)^{-1} \\ &= \sum_{j=0}^{\infty} (-\theta)^j L^j \end{aligned}$$

and consequently we can write

$$\begin{aligned} \sum_{j=0}^{\infty} (-\theta)^j L^j (Y_t - \mu) &= \epsilon_t \\ (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + ...)(Y_t - \mu) &= \epsilon_t \end{aligned}$$

which is an infinite AR process.

Invertability of the MA(q) process

If an MA(q) process can be given an infinite order AR process, then the MA(q) process is said to be invertible.

Comment:

If $\theta(z)^{-1}$ exists, then $Y_t - \mu = \theta(L)\epsilon_{t,}$ is an invertible MA(q) process.

- Question: What does invertability means in terms of first and second order moments of the MA(q) process?
- To answer this question we need a "new" tool: The Autocovariance Generating Function ($g_Y(z)$).

Invertability of the MA(q) process

The autocovariance generating function is defined as

$$g_Y(z) = \sum_{j=-\infty}^{j=\infty} \gamma_j z^j$$

where z is a scalar.

- Why useful: If two stochastic processes have identical autocovariance generating functions, then the two processes exhibit identical sequences of autocovariances.

Proposition:
For any invertible MA(q) representation, there exists a non-invertible MA(q) representation with identical first and second order moments.

Proof: Hamilton, pages 64–66

Invertability of the MA(q) process

- The implication is that the invertible or the non-invertible MA(q) process could characterize data equally well (in terms of the first two moments).
- However, to find ϵ_t for date t associated with the invertible MA(q), we need only to know the current and past values of Y_t. This implies that ϵ_t can be computed using real world-data.
- To find $\widetilde{\epsilon}_t$ associated with an non-invertible MA(q) we need to know all the future values of Y_t

Invertability of the MA(q) process

Comment:

- The invertible representation should "always" be preferred.
- The values of ϵ_t associated with invertible MA(q) representation are called the Fundamental Innovation for Y_t.

Example

$\tilde{Y}_t = (1 + 2.4L + 0.8L^2)\tilde{\epsilon}_t, \quad \epsilon_t \sim N(0, \tilde{\sigma}^2)$ is not invertible because

```
> polyroot(c(1, 2.4, 0.8))

[1] -0.5-0i -2.5+0i
```

$$(1 + 2.4z + 0.8z^2) = (1 - \frac{1}{(-0.5)}z)(1 - \frac{1}{(-2.5)}z)$$

- $z_1 = -0.5$ is inside the unit circle (no invertibility)
- None are in the circle (we can find an invertible process with the same moments)

The inverse roots $\lambda_1 = 1/(-2.5) = -0.4 \quad \lambda_2 = 1/(-0.5) = -2$

$$\tilde{Y}_t = (1 - \lambda_1 L)(1 - \lambda_2 L)\epsilon_t$$

Example

The invertible process:

$$\begin{aligned} Y_t =& (1-(-0.4)L)(1-(-0.5)L)\epsilon_t, \quad \epsilon_t \sim N(0, \tilde{\sigma}^2) \\ =& (1-\lambda_1 L)(1-\lambda_2^{-1} L)\epsilon_t \\ =& (1+0.9L+0.2L^2)\epsilon_t \end{aligned}$$

for $\sigma^2 = 4\tilde{\sigma}^2$, is invertible and has the same moments than the intial process.

Note: We only use λ_2 because it is outside the unit circle.
Remark: The roots of the final polynomial are all outside the unit circle.