GARCH Model

- The ARCH model often requires a large order to adequately describe the volatility process of returns.
- Bollerslev proposes a Generalised ARCH model
- ARCH is like an AR model for volatility
- GARCH is like an ARMA model for volatility.

GARCH Model

The GARCH(p, q) model is:

$$
\begin{aligned}
r_t &= \mu_t + \epsilon_t \\
\epsilon_t &= z_t \sigma_t, \qquad z_t \sim IID(0,1) \\
\sigma_t^2 &= a_0 + a(L)\epsilon_t^2 + b(L)\sigma_t^2, \qquad a_0 > 0 \\
a(L) &= a_1 L + \ldots + a_p L^p, \quad a_i \geq 0 \\
b(L) &= b_1 L + \ldots + b_q L^q \quad b_j \geq 0
\end{aligned}
$$

Note: for identification of b_j, must have at least one ARCH coefficient $a_i > 0$

Conditional Mean Specification

- $\mu_t = E[r_t | F_{t-1}]$ is typically specified as a constant or possibly a low order ARMA process to capture autocorrelation caused by market microstructure effects (e.g., bid-ask bounce) or non-trading effects.
- If extreme or unusual market events have happened during sample period, then dummy variables associated with these events are often added to the conditional mean specification to remove these effects. The typical conditional mean specification is of the form

$$E[r_t | F_{t-1}] = c + \sum_{i=1}^{p} \phi_i y_{t-i} + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} + \sum_{l=0}^{L} \beta_l' \mathbf{x}_{t-l} + \epsilon_t$$

where $\mathbf{x}_t$ is a $k \times 1$ vector of exogenous explanatory variables.

Explanatory Variables in the Conditional Variance Equation

- Exogenous explanatory variables may also be added to the conditional variance formula

$$\sigma_t^2 = a_0 + \sum_{i=1}^{p} a_i \epsilon_{t-i}^2 + \sum_{j=1}^{q} b_j \sigma_{t-j}^2 + \sum_{k=1}^{K} \delta'_k \mathbf{y}_{t-k}$$

  where $\mathbf{y}_t$ is a $m \times 1$ vector of variables, and δ is a $m \times 1$ vector of positive coefficients.

- Variables that have been shown to help predict volatility are: trading volume, interest rates, macroeconomic news announcements, implied volatility from option prices and realized volatility, overnight returns, and after hours realized volatility

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Properties of GARCH model

- Let a random variable $\eta_t = \epsilon_t^2 - \sigma_t^2 = \epsilon_t^2 - E(\epsilon_t^2 | F_{t-1})$.
- It can be shown that this variable is a Martingale difference series: $E(\eta_t) = 0$ and it is uncorrelated. However it is not indenpendent.
- Substituying $\sigma_t^2 = \epsilon_t^2 - \eta_t$ in the GARCH equation we have:

$$\epsilon_t^2 = a_0 + \sum_{i=1}^{\max(p,q)} (a_i + b_i)\epsilon_{t-i}^2 + \eta_t - \sum_{j=1}^{q} b_j \eta_{t-j}$$

Q: Does it remind you of any process?

Properties of GARCH model

The GARCH(p,q) is an ARMA(max(p,q),q) of ϵ_t^2

- What is its unconditional mean of ϵ_t^2?
- What is the unconditional variance of ϵ_t?
- Is ϵ_t weakly stationary?

Properties of GARCH model

1. GARCH(p, q) is equivalent to ARCH(∞). If $1 - b(z) = 0$ has all roots outside unit circle then

$$\begin{aligned}
\sigma_t^2 =& \frac{a_0}{1 - b(1)} + \frac{a(L)}{1 - b(L)} \epsilon_t^2 \\
=& a_0^* + \delta(L)\epsilon_t^2, \qquad \delta(L) = \sum_{k=0}^{\infty} \delta_k L^k
\end{aligned}$$

2. ϵ_t is a stationary and ergodic with finite variance provided $a(1) + b(1) < 1$

$$\begin{aligned}
E[\epsilon_t] =& 0 \\
var(\epsilon_t) =& E[\epsilon_t^2] = \frac{a_0}{1 - a(1) - b(1)}
\end{aligned}$$

$$\epsilon_t^2 \sim ARMA(m, q) \quad m = \max(p, q)$$

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GARCH(1,1)

The most commonly used GARCH(p,q) model is the GARCH(1,1)

$$\sigma_t^2 = a_0 + a_1\epsilon_{t-1}^2 + b_1\sigma_{t-1}^2$$

Show the following properties:

$$\begin{aligned}\text{stationarity condition:}&\\ \text{ARCH}(\infty):&\\ \text{ARMA}(1,1):&\\ \text{unconditional variance:}&\end{aligned}$$

GARCH(1,1)

- Large ϵ_{t-1}^2 implies large σ_t^2 as in the ARCH but also does large σ_{t-1}^2
- Large innovations tend to be followed by large innovations (volatility clustering)
- Leptokurtosis: if $1 - 2a_1^2 - (a_1 + b_1)^2 > 0$ then

$$\frac{E(\epsilon_t^4)}{E^2(\epsilon_t^2)} = \frac{3(1 - (a_1 + b_1)^2)}{1 - 2a_1^2 - (a_1 + b_1)^2} > 3$$

- This means that the tail distribution of a GARCH(1,1) process is heavier than that of a normal distribution
- The model describes how the volatility evolves

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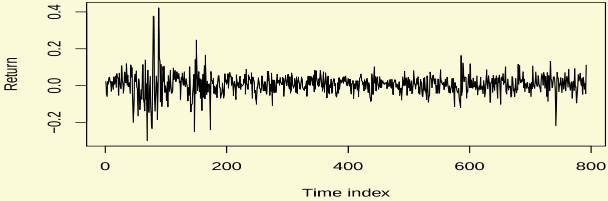
GARCH model

- Specifying the order of a GARCH model is not easy.
- In practice, only small order models are used: GARCH(1,1), GARCH(2, 1) and GARCH(1,2).
- The conditional maximum likelihood method contine to apply provided that the starting values of σ_t^2 are known.
- For example, if σ_1^2 is treated as fixed, then σ_t^2 can be computed recursively for a GARCH(1,1) model
- In some applications, the sample variace of ϵ_t serves as a good starting value of σ_1^2
- The fitted model can be checked by using the standarised residuals $\hat{\epsilon}$ and its squared process

Example

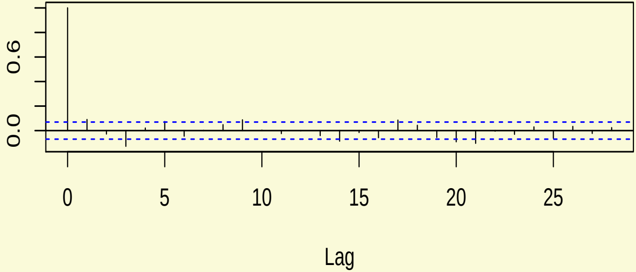
We consider the montly excess returns (excess returns are generally defined as the returns provided by a given portfolio minus the returns provided by a risk-free asset) of S&P 500 from 1926 ($T=$ 792). The file is sp500.dat.

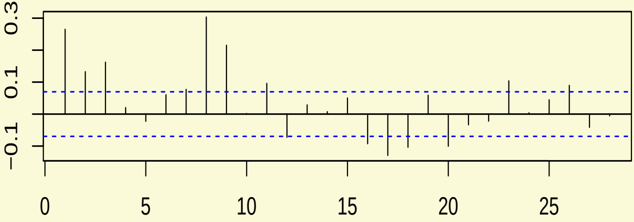
```
> sp500=scan("../data/sp500.dat")
> plot(sp500, type="l",ylab="Return", xlab="Time index")
```



Example

Top to bottom: ACF of returns, PACF of squared returns





Example

- From the ACF, we see some serial correlation in the returns at lags 1 and 3 (MA(3))
- The PACF of r_t^2 shows strong linear dependence (heteroskedasticity)
- We fit an MA(3) model without second lag

```
> sp500.ma<-arima(sp500, order=c(0,0,3),
+                  fixed= c(NA, 0, NA, NA))
> theta<-round(coef(sp500.ma),4)
```

$$r_t = 0.0062 + \epsilon_t + 0.094\epsilon_{t-1} - 0.1407\epsilon_{t-3} \quad \hat{\sigma}^2 = 0.0576$$

Example

In this example we use an AR(3) which is more simple.

```
> sp500.ar<-arima(sp500, order=c(3,0,0), method="ML")
> phi<-round(coef(sp500.ar),4)
> phi

     ar1       ar2       ar3 intercept
  0.0890   -0.0238   -0.1229    0.0062

> epsilon_t=residuals(sp500.ar)
```

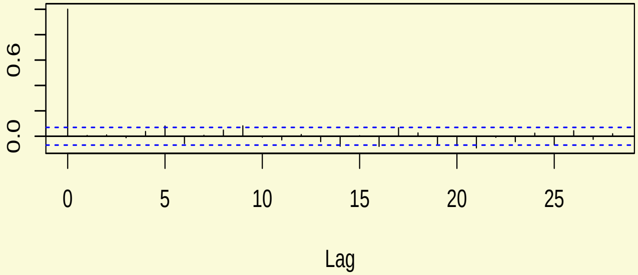
$r_t = 0.0062 + 0.089 r_{t-1} - 0.0238 r_{t-2} - 0.1229 r_{t-3} + \epsilon_t \quad \hat{\sigma}^2 = 0.0033$

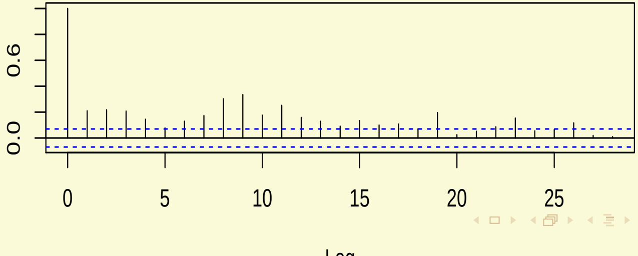
Plot the ACF of the residuals and squared residuals.

$$\epsilon_t = r_t - \mu_t = \sigma_t z_t$$

We see that the residuals are uncorrelated but there is correlation in the squared residuals $\Rightarrow$ heteroskedasticity

Example





Example

We want to fit a GARCH(1,1) on the residuals

$$
\begin{aligned}
\epsilon_t &= \sigma_t z_t \\
\sigma_t^2 &= a_0 + a_1 \epsilon_{t-1}^2 + b_1 \sigma_{t-1}^2
\end{aligned}
$$

We do a joint estimation of the AR(3)-GARCH(1,1) in R

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Example

```
> library(fGarch)
> r=epsilon_t^2
> sp500.garch<-garchFit(r~garch(1,1), trace=F,
+                       include.mean=F)
> coef<-coef(sp500.garch)
> coef

    omega    alpha1     beta1
0.0108681 0.1543255 0.8045162

> #Implied unconditional variance of epsilon_t
> sigma2.bar=coef[1]/(1-sum(coef[2:3]))
> sigma2.bar

    omega
0.2640563
```

Run the command summary(sp500.garch) at home.

Example

One step command:

```
> sp500.garch2<-garchFit(sp500~arma(3,0)+garch(1,1), trace=F)
> coef2<-coef(sp500.garch2)
> round(coef2,4)

     mu      ar1      ar2      ar3    omega   alpha1    beta1
-0.0060   0.0536  -0.0280   0.0171   0.0115   0.1602   0.7958

> #Implied unconditional variance of epsilon_t
> sigma.bar.2=coef2[5]/(1-sum(coef2[6:7]))
> sigma.bar.2

   omega
0.261744
```

GARCH-in-Mean (GARCH-M)

Idea: Modern finance theory suggests that volatility may be related to risk premia on assets.

The GARCH-M model allows time-varying volatility to be realted to expected returns

$$r_t = c + \alpha g(\sigma_t) + \epsilon_t \quad \epsilon_t \sim GARCH$$

Choices of $g(\sigma_t)$:

- σ_t
- σ_t^2
- $\ln(\sigma_t^2)$

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Temporal Aggregation

- Volatility clustering and non-Gaussian behavior in financial returns is typically seen in weekly, daily or intraday data. The persistence of conditional volatility tends to increase with the sampling frequency.
- For GARCH models there is no simple aggregation principle that links the parameters of the model at one sampling frequency to the parameters at another frequency. This occurs because GARCH models imply that the squared residual process follows an ARMA type process with MDS innovations which is not closed under temporal aggregation.

Temporal Aggregation

- The practical result is that GARCH models tend to be fit to the frequency at hand. This strategy, however, may not provide the best out-of-sample volatility forecasts. For example,Martens (2002) showed that a GARCH model fit to S&P 500 daily returns produces better forecasts of weekly and monthly volatility than GARCH models fit to weekly or monthly returns, respectively.