Introduction

(Hamilton: Chapters 1 and 2)

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Introduction to Time Series

- Definition of time series
- Dynamic multipliers
- Difference equations
- Lag operators

What are dynamic models?

Difference between regressions and dynamic models:

- Simple linear regression, given the r.v $\{Y_i\}_{i=1}^{N}$ and regressors X_i

$$y_i = X_i\beta + \epsilon_i$$

  where

  - $\epsilon_i \sim IID(0, \sigma^2)$, $i = 1, 2, \ldots, N$

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Linear regression

Joint density of $\{Y_i\}_{i=1}^N$

$$f_Y(y_1, \ldots, y_N; \beta, \sigma^2) = \prod_{i=1}^N f_i(y_i; \beta, \sigma^2)$$

Joint density of $\epsilon_i = y_i - X_i\beta$

$$f_\epsilon(\epsilon_1, \ldots, \epsilon_N; \sigma^2) = \prod_{i=1}^N f_{\epsilon_i}(\epsilon_i; \sigma^2)$$

The ordering of observations does not matter.

Dynamic model

Simple dynamic (autoregressive) model:

$$Y_t = \phi Y_{t-1} + \epsilon_t$$

where $\epsilon_t \sim IID(0, \sigma^2), t = 1, 2, \ldots, T$

Joint density of $\{Y_t\}_{t=1}^{T}$:

$$f_Y(y_1, \ldots, y_T; \beta, \sigma^2) \neq \prod_{t=1}^{T} f_t(y_t; \beta, \sigma^2)$$

unless $\phi = 0$.

The observations are correlated and ordered. The ordering matters.

Time series

Everything measure in time is special:

- Learn of your own mistakes — the past have information

Time series

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- History repeats itself, first as tragedy, second as farce — Karl Marx
  - Seasonality, long memory, stationarity

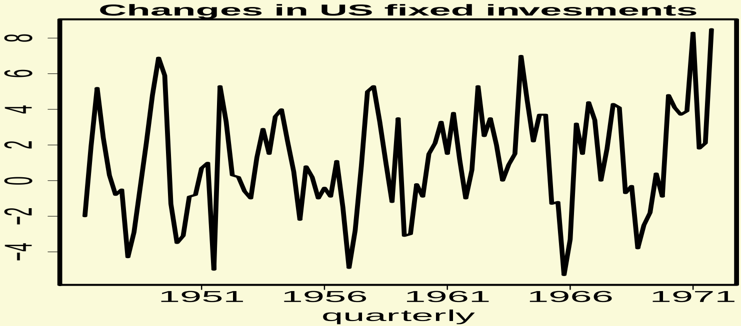
Time series

Everything measure in time is special:

- Learn of your own mistakes — the past have information
- History repeats itself, first as tragedy, second as farce — Karl Marx
  - Seasonality, long memory, stationarity
- Time is a drug. Too much of it kills you – Terry Pratchett

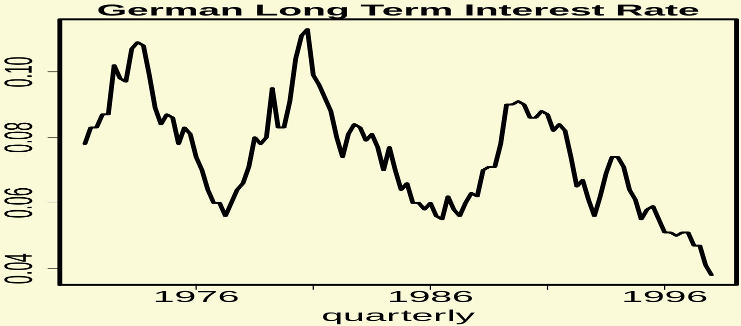
So, a process recorded over time is special. Therefore we separate time series from microeconometrics.

Real examples



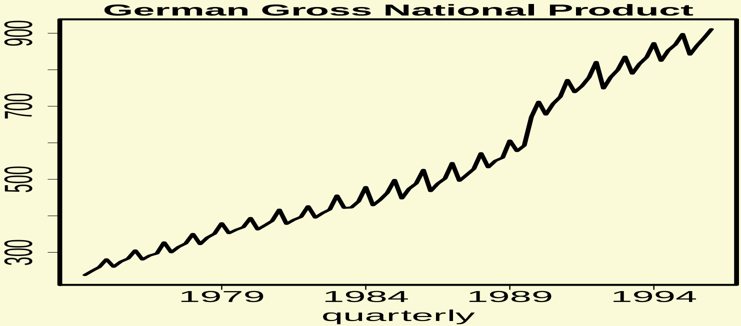
- Fluctuates randomly around a constant mean
- Variability is homogenous, independent of time

Real examples



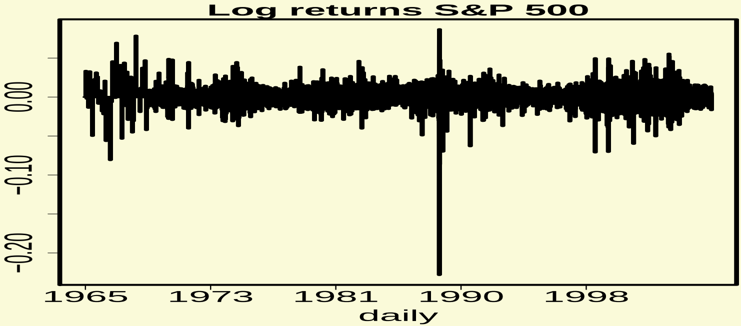
- There are not as many fluctuations
- Regular variability

Real examples



- Evolves following a polynomial trend
- Seasonality
- Level shift around 1990

Real examples



- Moves around a fixed mean
- Variability is heterogenous, it clusters.
- Unusual spike October, 19 1987 — Black Monday. Some say that it is an outlier.

Real examples

- Different series have different properties: seasonality, level shifts, etc
- These have to be taken into account to find the appropriate model, DGP (Data Generating Process)
- A stochastic process is a collection of random variables with a time index $\{Y_t\}, t \in \mathbb{T}$
- A sample generated from this stochastic process is also named time series $y_1, \ldots, y_T$

Formal definition

A *random process/ stochastic process/ random function* is a collection $\{ Y_t : t \in \mathbb{T}\}$ of random variables defined on some common probability space (sample space + probability function) and taking values in some state space, usually $\mathbb{R}$ or some subset. Usually the index set $\mathbb{T}$ will be either:

- $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = [0, \infty)$ (continuous time)
- $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \{0, 1, 2, \ldots\}$ (discrete time)

When the index represents time, then we say that $\{ Y_t, t \in \mathbb{T}\}$ is a *time series*.

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Time series

- Basically, a time series is a stochastic process which is observed over time.
- Therefore, it is defined by specifying joint distributions for all posible finite subcollections.
- The joint distributions must satisfy certain consistency in the sense that the marginal should be recovered.
- It can be a continuous time process
- or a discrete time process

First-order difference equations

$$y_t = \phi y_{t-1} + w_t$$

- The above equation is a *linear first-order difference equation*
- It can also be written as:

$$y_t - y_{t-1} = (\phi - 1) y_{t-1} + w_t \quad (1)$$

$$\Delta y_t = (\phi - 1) y_{t-1} + w_t \quad (2)$$

where $t = 1, \ldots, T$, $y_t, w_t \in \mathbb{R}$ and ϕ a scalar

First-order difference equations

$$y_t = \phi y_{t-1} + w_t, \qquad t > 0 \tag{3}$$

Examples:

1. $w_t = \mathbf{x}_t\beta$, $\beta = (\beta_1, ..., \beta_p)'$; $\mathbf{x}_t = (x_{1t}, ..., x_{pt})'$, fixed
2. $w_t = \delta t$
3. $w_t = \varepsilon_t$, $\{\varepsilon_t\} \sim \mathrm{N}(0, \sigma^2)$.
4. $w_t = c$ (constant).

Case 3 is an autoregressive model of order one [AR(1) model] with no intercept.

Example difference equations

Goldfeld (1973) estimated the USA money demand at time t.

He assumes that the log of the real money holdings of the public, m_t, depends on:

- The first lag of the money demand: m_{t-1}
- The log of the aggregate real income at time t: I_t
- The log of the interest rate on bank accounts at time t: r_{bt}
- The log of interest rate on commercial paper at time t: r_{ct}
- The model: $m_t = \phi m_{t-1} + w_t$. In particular:

$$m_t = 0.27 + 0.72 m_{t-1} + 0.19 I_t - 0.045 r_{bt} - 0.019 r_{ct}$$

Difference equations

Main questions of interest:

- Is the process y_t **stable** or **explosive**?
- What are the effects on the future path of y_t of changes (permanent and/or transitory) in w_t?

To answer the questions we need to solve the difference equation.

We can solve them by

- Simple iterations.
- Lag operators.

Solving by iteration

Solution by iteration:

$$y_t = \phi y_{t-1} + w_t$$

$$\begin{aligned}
y_0 &= \phi y_{-1} + w_0, \\
y_1 &= \phi y_0 + w_1 = \phi(\phi y_{-1} + w_0) + w_1 = \phi^2 y_{-1} + \phi w_0 + w_1, \\
y_2 &= \phi y_1 + w_2 \\
&= \phi^3 y_{-1} + \phi^2 w_0 + \phi w_1 + w_2
\end{aligned}$$

Solving by iteration

The general solution:

$$
\begin{aligned}
y_t &= \phi^{t+1} y_{-1} + \phi^t w_0 + \phi^{t-1} w_1 + \ldots + \phi w_{t-1} + w_t, \\
&= \phi^{t+1} y_{-1} + \sum_{i=0}^{t} \phi^{t-i} w_i
\end{aligned} \tag{4}
$$

Remark 1: Once we know the past (historical values), we can estimate the future

Remark 2: Values y_{-1} and $w_0, \ldots, w_t$ are known

Solving by iteration

The above calculations would be identical if the recursions where started at period t with y_{t-1} as given, i.e.,

$$\begin{aligned} y_{t+j} &= \phi^{j+1} y_{t-1} + \phi^j w_t + \phi^{j-1} w_{t+1} + \ldots + \phi w_{t+j-1} + w_{t+j}, \\ &= \phi^{j+1} y_{t-1} + \sum_{i=0}^{j} \phi^{j-i} w_{t+i} \qquad (5) \end{aligned}$$

y_{t+j} is a linear function in y_{t-1} and the values of w_{t+i}. The parameters ϕ^j are the **dynamic multipliers**.

Def: **Dynamic multipliers** is the effect of w_t on y_{t+j}.

Dynamic multipliers

$$y_t = \phi^{t+1} y_{-1} + \phi^t w_0 + \phi^{t-1} w_1 + \ldots + \phi w_{t-1} + w_t$$

How does y_t gets affected when w_0 changes if y_{-1} and $w_1, \ldots, w_t$ are kept unchanged?:

$$\frac{\partial y_t}{\partial w_0} = \phi^t$$

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Dynamic multipliers

$$y_{t+j} = \phi^{j+1} y_{t-1} + \phi^j w_t + \phi^{j-1} w_{t+1} + \ldots + \phi w_{t+j-1} + w_{t+j}$$

How does y_{t+j} gets affected when w_t changes if y_{t-1} and $w_t, \ldots, w_{t+j}$ are kept unchanged?:

$$\frac{\partial y_{t+j}}{\partial w_t} = \phi^j$$

- The dynamic multiplier depends only on the lag length j, the effect does not change with t
- ϕ^j (as a function of j) is also called the *impulse response function*
- The impulse response function characterises the effect of a single impulse or input w_t on y_{t+j}, $j = 1, 2, \ldots$

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Example dynamic multiplier

Goldfeld (1973)

$$m_t = 0.27 + 0.72 m_{t-1} + 0.19 I_t - 0.045 r_{bt} - 0.019 r_{ct}$$

- ϕ= 0.72
- $w_t = 0.27 + 0.19 I_t - 0.045 r_{bt} - 0.019 r_{ct}$

Q (3 minutes): What will happen to money demand (m_t) two quarters from now if current income I_t increases by one unit today (log scale$- >$ 1%) with future incomes I_{t+1}, I_{t+2} unaffected?

Example dynamic multiplier

Goldfeld (1973)

$$m_t = 0.27 + 0.72 m_{t-1} + 0.19 I_t - 0.045 r_{bt} - 0.019 r_{ct}$$

- $\phi = 0.72$
- $w_t = 0.27 + 0.19 I_t - 0.045 r_{bt} - 0.019 r_{ct}$

Q (3 minutes): What will happen to money demand (m_t) two quarters from now if current income I_t increases by one unit today (log scale$->$ 1%) with future incomes I_{t+1}, I_{t+2} unaffected?
We know that the equation above will be written as:

$$m_{t+2} = \phi^2 m_{t-1} + \phi^2 w_t + \phi w_{t+1} + w_{t+2}$$

Example dynamic multiplier

Then, the effect we are looking for is:

$$\begin{aligned}\frac{\partial m_{t+2}}{\partial I_t} =& \frac{\partial m_{t+2}}{\partial w_t} \times \frac{\partial w_t}{\partial I_t} \\ =& \phi^2 \times \beta_{I_t} = 0.72^2 \times 0.19 = 0.098\end{aligned}$$

Note, that the dynamic multiplier is not only the impulse response ϕ^2...

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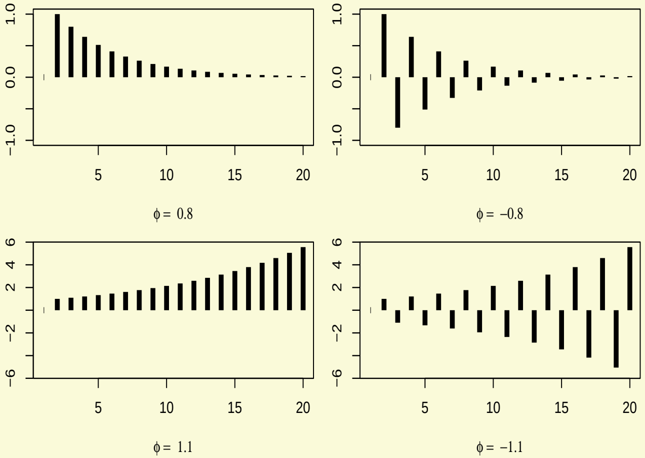
Dynamic multipliers

Coefficient ϕ produce a variety of dynamic responses (assume $w_t = 0$ for all t):

Six cases:

1. $0 < \phi < 1$, $\partial y_t / \partial w_t = \phi^t \to 0$ as $t \to \infty$, monotonically: dependence on y_{-1} gradually vanishes.
2. $-1 < \phi < 0$, $\phi^{t+1} y_{-1} \to 0$ as $j \to \infty$: dependence on y_{-1} oscillates with period 2 and gradually vanishes.
3. $\phi > 1$, $\phi^{t+1} y_{-1} \to \infty$ as $j \to \infty$: 'diverges' or 'explodes', y_t always dependent on y_{-1}
4. $\phi < -1$, $\phi^{t+1} y_{-1} \to \pm\infty$ as $t \to \infty$: 'diverges', oscillates with period 2, y_t always dependent on y_{-1},

Dynamic multipliers



Dynamic multipliers

Coefficient ϕ controls the nature of the response (assume $w_t = 0$ for all t):

Six cases:

5. $\phi = 1$, $\phi^{t+1} y_{-1} = y_{-1}$: all values in the sequence dependent on y_{-1}, no divergence.
6. $\phi = -1$, $\phi^{t+1} y_{-1} = (-1)^{t+1} y_{-1}$: all values in the sequence dependent on y_{-1}, no divergence.

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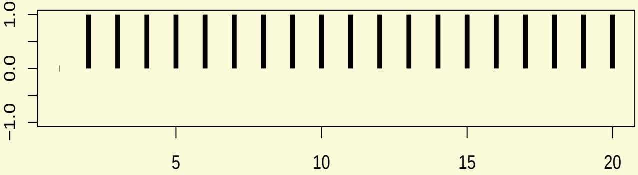
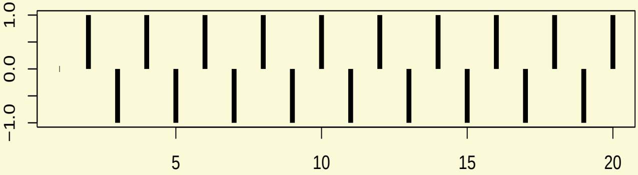
Dynamic multipliers



$\phi = 1$



$\phi = -1$

Dynamic multipliers

Summary:

- If $|\phi| < 1$ the system is stable $\rightarrow$ the effect on future values y_{t+j} of a transitory change in w_t will eventually die out as $j \rightarrow \infty$.
- If $|\phi| \geq 1$ the system is explosive $\rightarrow$ the effect on future values y_{t+j} of a transitory change in w_t will be permanent even as $j \rightarrow \infty$.

Permanent change in w

What are the dynamic consequences of a permanent change in w?

Not only w_t but also $w_{t+1}, w_{t+2}, ..., w_{t+j}$ all increase by one unit. This implies

$$\frac{\partial y_{t+j}}{\partial w_{t+j}} + \frac{\partial y_{t+j}}{\partial w_{t+j-1}} + ... + \frac{\partial y_{t+j}}{\partial w_{t+1}} + \frac{\partial y_{t+j}}{\partial w_t} = 1 + \phi + ... + \phi^{j-1} + \phi^j.$$

Permanent change in w

Assume $|\phi| < 1$.

$$\lim_{j\to\infty}\left(\frac{\partial y_{t+j}}{\partial w_{t+j}}+\frac{\partial y_{t+j}}{\partial w_{t+j-1}}+\ldots+\frac{\partial y_{t+j}}{\partial w_{t+1}}+\frac{\partial y_{t+j}}{\partial w_{t}}\right)=\lim_{j\to\infty}\sum_{k=0}^{j}\phi^{k}=\frac{1}{1-\phi}.$$

- This limit is often referred to as the long run effect of w on y
- If $|\phi| \geq 1$: The limit is not defined!

Cumulative effect of w_t change

The cumulative effects on y_{t+j} of a transitory change in w_t

$$\sum_{i=0}^{j} \frac{\partial y_{t+i}}{\partial w_t} = \frac{\partial y_t}{\partial w_t} + \frac{\partial y_{t+1}}{\partial w_t} + \ldots + \frac{\partial y_{t+j}}{\partial w_t} = 1 + \phi + \ldots + \phi^j.$$

If $|\phi| < 1$:

$$\lim_{j\to\infty} \sum_{i=0}^{j} \frac{\partial y_{t+i}}{\partial w_t} = \lim_{j\to\infty} \sum_{i=0}^{j} \phi^i = \frac{1}{1-\phi}.$$

Cumulative effect of w_t change

Summary:

The **cumulative effect** on y_{t+j} of a transitory change in w_t equals the **long run effect** on y_{t+j} of a permanent change in w_t.

Exercise (3 minutes)

- What is the cumulative effect over the money demand 10 quarters from now if the current income I_t were to increase by one unit but with future income I_{t+j} for $j = 1, \ldots, 10$ unchanged?
- What is the long run effect over the money demand 10 quarters from now if the incomes I_{t+j} were to increase by one unit?

pth-order difference equations

The pth-order difference equation has the form

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + w_t. \tag{6}$$

Q: Solve a 2nd-order difference equation by iteration? How many initial conditions do you need?

pth-order difference equations

In order to simplify the computations of dynamic multipliers it is often convenient to rewrite (6) as:

$$\underset{(p\times 1)}{\xi_t} = \underset{(p\times p)}{\mathbf{F}}\,\underset{(p\times 1)}{\xi_{t-1}} + \underset{(p\times 1)}{\mathbf{v}_t}, \tag{7}$$

where

$$\begin{aligned}
\xi_t &= (y_t, y_{t-1}, \ldots, y_{t-p+1})', \\
\mathbf{F} &= \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \\
\mathbf{v}_t &= (w_t, 0, \ldots, 0)'.
\end{aligned}$$

Example: third-order difference equation

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + w_t$$

$$\begin{bmatrix} y_0 \\ y_{-1} \\ y_{-2} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{-1} \\ y_{-2} \\ y_{-3} \end{bmatrix} + \begin{bmatrix} w_0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xi_{\mathbf{0}} = \mathbf{F}\xi_{\mathbf{-1}} + \mathbf{v_0}$$

Note: 3-rd order $\Rightarrow$ three initial conditions

pth-order difference equations

$$
\begin{aligned}
\xi_{\mathbf{0}} =& \mathbf{F}\xi_{-\mathbf{1}} + \mathbf{v_0}, \\
\xi_{\mathbf{1}} =& \mathbf{F}\xi_{\mathbf{0}} + \mathbf{v_1} = \mathbf{F}^2\xi_{-\mathbf{1}} + \mathbf{F}\mathbf{v_0} + \mathbf{v_1}, \\
\vdots =& \vdots \\
\xi_{\mathbf{t}} =& \mathbf{F}^{t+1}\xi_{-\mathbf{1}} + \mathbf{F}^{t}\mathbf{v_0} + \ldots + \mathbf{F}\mathbf{v_{t-1}} + \mathbf{v_t},
\end{aligned}
$$

That is:

$$
\xi_{\mathbf{t}} = \mathbf{F}^{t+1}\xi_{-\mathbf{1}} + \sum_{i=0}^{t} \mathbf{F}^{i}\mathbf{v}_{t-i}
$$

pth-order difference equations

- Note that (7) is a system of p equations. The first equation equals (6) whereas the remaining equations are merely trivial identities.
- Based on the representation given by (7) we can now - as previously described - characterize its solution as

$$\xi_{t+j} = \mathbf{F}^{j+1}\xi_{t-1} + \sum_{i=0}^{j} \mathbf{F}^{j-i}\mathbf{v}_{t+i}, \tag{8}$$

where the first element (the element of interest) in ξ_{t+j} can be found as

$$\begin{aligned} y_{t+j} \quad = \quad & f_{11}^{(j+1)} y_{t-1} + f_{12}^{(j+1)} y_{t-2} + \ldots + f_{1p}^{(j+1)} y_{t-p} \\ & + f_{11}^{(j)} w_t + f_{11}^{(j-1)} w_{t+1} + \ldots + f_{11}^{(1)} w_{t+j-1} + w_{t+j} \end{aligned} \tag{9}$$

where $f_{il}^{(j)}$ denotes the (i, l) entry in $\mathbf{F}^j$.

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Example: third-order dynamic multipliers

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + w_t$$

$$\begin{bmatrix} y_0 \\ y_{-1} \\ y_{-2} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{-1} \\ y_{-2} \\ y_{-3} \end{bmatrix} + \begin{bmatrix} w_0 \\ 0 \\ 0 \end{bmatrix}$$

$$\partial y_{t+j} / \partial w_t = f_{11}^{(j)}$$

so

$$\begin{aligned} \frac{\partial y_{t+1}}{\partial w_t} &= \phi_1 \text{ from } \mathbf{F}, \\ \frac{\partial y_{t+2}}{\partial w_t} &= \phi_1^2 + \phi_2, \text{ from } \mathbf{F}^2, \end{aligned}$$

etc.

Example: third-order dynamic multipliers

$$\xi_1 = \mathbf{F}^2 \xi_{\mathbf{-1}} + \mathbf{F}\mathbf{v_0} + \mathbf{v_1}$$

where

$$\mathbf{F}^2 = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \phi_1^2 + \phi_2 & \phi_1\phi_2 + \phi_3 & \phi_1\phi_3 \\ \phi_1 & \phi_2 & \phi_3 \\ 1 & 0 & 0 \end{bmatrix}$$

Q: What is $\frac{\partial y_{t+2}}{\partial w_{t+1}}$?

Q: What is $\frac{\partial y_{t+3}}{\partial w_t}$?

Example: third-order dynamic multipliers

$$\xi_1 = \mathbf{F}^2 \xi_{-\mathbf{1}} + \mathbf{F}\mathbf{v_0} + \mathbf{v_1}$$

where

$$\mathbf{F}^2 = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \phi_1^2 + \phi_2 & \phi_1\phi_2 + \phi_3 & \phi_1\phi_3 \\ \phi_1 & \phi_2 & \phi_3 \\ 1 & 0 & 0 \end{bmatrix}$$

Q: What is $\frac{\partial y_{t+2}}{\partial w_{t+1}}$?
Q: What is $\frac{\partial y_{t+3}}{\partial w_t}$?

Note: We always use the first element of $\mathbf{F}^j$.

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Effect of transitory change in w_t

- A transitory change in w_t : From (8)

$$\frac{\partial \xi_{t+j}}{\partial \mathbf{v}_t'} = \mathbf{F}^j,$$

hence

$$\frac{\partial y_{t+j}}{\partial w_t} = f_{11}^{(j)},$$

which can also easily be verified from (9).

- A closed form expression for $f_{11}^{(j)}$ can be obtained. Will depend entirely on the **eigenvalues** of $\mathbf{F}$.

Cumulative effect of a transitory change

- The cumulative effects of a transitory change in w_t : From (8)

$$\sum_{j=0}^{\infty} \frac{\partial \xi_{t+j}}{\partial \mathbf{v}'_t} = \sum_{j=0}^{\infty} \mathbf{F}^j = (\mathbf{I}_p - \mathbf{F})^{-1},$$

  This sum is less than ∞ if the eigenvalues of $\mathbf{F}$ are all inside the unit circle (meaning that the absolute value or modulus of the eigenvalues are less than unity).

- Recall: The eigenvalues of a matrix $\mathbf{F}$ are those numbers λ for which

$$\det(\mathbf{F} - \lambda \mathbf{I}_p) = 0.$$

Dynamic multipliers (pth order difference equation)

Proposition: The eigenvalues of the matrix $\mathbf{F}$ in (7) are the values of λ that satisfies

$$\lambda^p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \ldots - \phi_{p-1}\lambda - \phi_p = 0.$$

So find the roots of this polynomial and check their modulus to see whether they are inside or outside the unit circle.

If so, the cumulative effect of a transitory change will converge to a finite value

Dynamic multipliers (pth order difference equation)

Given that $(\mathbf{I}_p - \mathbf{F})^{-1}$ exists its (1,1) element is given by

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{\partial y_{t+j}}{\partial w_t} &= \frac{1}{1-\phi_1-\phi_2-\ldots-\phi_p}, \\ &= \lim_{j\to\infty}\left(\frac{\partial y_{t+j}}{\partial w_t}+\frac{\partial y_{t+j}}{\partial w_{t+1}}+\ldots+\frac{\partial y_{t+j}}{\partial w_{t+j}}\right),\end{aligned}$$

Again we have that the cumulative effect of the change of w_t is equal to the effect of permanent change of w.

Dynamic multipliers (pth order difference equation)

Summary:

In the pth order difference equation represented by

$$\underset{(p\times1)}{\xi_t} = \underset{(p\times p)}{\mathbf{F}}\,\underset{(p\times1)}{\xi_{t-1}} + \underset{(p\times1)}{\mathbf{v}_t}\,, \tag{10}$$

all the dynamic behaviour is determined by the eigenvalues of $\mathbf{F}$.

Eigenvalues

- All eigenvalues are in the unit circle (modulus greater less than one) $\implies$ stationary solution (the impulse response converges to zero as $j \to \infty$).
- At least one eigenvalue on the unit circle, the remaining ones inside $\implies$ no stationary solution (an impulse has a lasting effect).
- At least one eigenvalue outside the unit circle $\implies$ the process is explosive (the effect of an impulse increases infinitely over time).

Roots of the polynomial

The dynamic behaviour of y_t in the pth order difference equation can also be expressed by the roots of a polynomial:

$$y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \ldots - \phi_p y_{t-p} = w_t \tag{11}$$

is characterized by the roots of

$$1 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p = 0$$

The roots of this polynomial are the inverse of the eigenvalues.

- All roots outside the unit circle (modulus greater than one) $\Longrightarrow$ stationary solution (the impulse response converges to zero as $j \to \infty$).
- At least one root on the unit circle, the remaining ones outside $\Longrightarrow$ no stationary solution (an impulse has a lasting effect).
- At least one root inside the unit circle $\Longrightarrow$ the process is explosive (the effect of an impulse increases infinitely over time).

Example: second-order

How are the solutions of:

$$y_t = 0.5y_{t-1} - 0.8y_{t-2} + w_t$$

First method. Find the eigenvalues of

$$\mathbf{F} = \begin{bmatrix} 0.5 & -0.8 \\ 1 & 0 \end{bmatrix}$$

i.e

$$\det(\mathbf{F} - \lambda I_2) = 0 \quad \Rightarrow \lambda^2 + 0.5\lambda + 0.8 = 0$$

Example: second-order

We get two eigenvalues:

$$\lambda_1 = 0.25 + 0.86i$$
$$\lambda_2 = 0.25 - 0.86i$$

In R:

```
> lambda <- polyroot(c(0.8, 0.5, 1))
> round(lambda, 2)
[1] -0.25+0.86i -0.25-0.86i
> abs(lambda)
[1] 0.8944272 0.8944272
```

Both have modulus $= 0.9$, both inside the unit circle. Therefore, the solutions are stationary.
It can be shown that because they are different the process will oscillate

Example: second-order

$$y_t = 0.5y_{t-1} - 0.8y_{t-2} + w_t$$

Second method. Find the roots of the polynomial

$$1 - 0.5z + 0.8z^2 = 0$$

Q: What do you expect?

$$z_1 = \lambda_1^{-1} = 0.25 - 0.86i$$
$$z_2 = \lambda_2^{-1} = 0.25 + 0.86i$$

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Example: second-order

In R:

```
> z <- polyroot(c(1, -0.5, 0.8))
> round(z, 2)

[1] 0.31+1.07i 0.31-1.07i

> abs(z)

[1] 1.118034 1.118034

> round(1/z, 2)

[1] 0.25-0.86i 0.25+0.86i

> abs(1/z)

[1] 0.8944272 0.8944272
```

Lag Operators

Time series lag operators

- Let $y_{t_1}, y_{t_2}, \ldots, y_{t_n}$ be a time series (for example: monthly Denmark GDP).
- This series is observer at n sucessive times $t_1, t_2, \ldots, t_n \in \mathbf{R}$
- The time series is a function that assigns a real number to each time t
- The *lag* (or *backward shift*) operator is denoted by L (or by B). It shifts a time series by one lag:

$$Ly_t = y_{t-1}$$

$$Ly_{t_i} = y_{t_{i-1}}$$

Exercises (3 minutes)

- What is $L(Ly_t)$?
- What is $L^2 y_t$?
- What is $L^s y_t$?
- What is $L^0 y_t$?
- What is $L^{-2} y_t$ (lead operator)?
- What is $y_t = \beta L y_t$?
- What is $y_t = L \beta y_t$?
- What is $L(y_t + x_t)$?
- What is $y_t = L(a + b y_t)$?
- What is $y_t = (a + bL) L x_t$?
- What is $(1 - \lambda_1 L)(1 - \lambda_2 L) x_t$?

Time series lag operators

For a function $f(y_t)$

$$Lf(y_t) = f(Ly_t) = f(y_{t-1})$$

However,

$$Lf(y_t) = f_{t-1}(y_t)$$

Polynomial lag operator

$$a(L) = 1 + a_1 L + a_2 L^2 + \ldots + a_p L^p$$

So,

$$(1 + a_1 L + a_2 L^2 + \ldots + a_p L^p) y_t = y_t + a_1 y_{t-1} + a_2 y_{t-1} + \ldots + a_p y_{t-p}$$

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Difference equations with lag operators

- Lag operators can be used to write difference equations

$$
\begin{aligned}
y_t &= \phi y_{t-1} + w_t \\
y_t &= \phi L y_t + w_t \\
y_t - \phi L y_t &= w_t \\
(1 - \phi L) y_t &= w_t
\end{aligned}
$$

What is $(1 - \phi L)^{-1}$?

Difference equations with lag operators

Multiply both sides of $(1-\phi L)y_t = w_t$ by $1+\phi L+\phi^2 L^2+\ldots+\phi^t L^t$:

$$
\begin{aligned}
(1+\phi L+\phi^2 L^2+\ldots+\phi^t L^t)(1-\phi L)y_t =& (1+\phi L+\phi^2 L^2+\ldots+\phi^t L^t)w_t \\
(1-\phi^{t+1}L^{t+1})y_t =& w_t+\phi w_{t-1}+\phi^2 w_{t-2}+\ldots+\phi^t w_0 \\
y_t =& \phi^{t+1}y_{-1}+w_t+\phi^2 w_{t-1}+\phi^2 w_{t-2}+\ldots+\phi^t w_0
\end{aligned}
$$

The solution is the same than the solution with the iterative method.

Q: What happens if $|\phi|<1$ and t is large?

Difference equations with lag operators

If $|\phi| < 1$ and t is large, then

$$(1 + \phi L + \phi^2 L^2 + \ldots + \phi^t L^t)(1 - \phi L)y_t = (1 - \phi^{t+1} L^{t+1})y_t \approx y_t$$

This means that

$$(1 + \phi L + \phi^2 L^2 + \ldots + \phi^t L^t)(1 - \phi L) \approx 1$$

i.e

$$\lim_{t\to\infty}(1 + \phi L + \phi^2 L^2 + \ldots + \phi^t L^t) = (1 - \phi L)^{-1}$$

inverse operator of $(1 - \phi L)$

Difference equations with lag operators

Remember that the inverse operator of $(1-\phi L)$ only exists if $|\phi|<1$:

$$\begin{aligned}(1-\phi L)y_t = w_t \iff y_t =&(1-\phi L)^{-1}w_t \\ =&w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \ldots + \phi^t w_0\end{aligned}$$

n-order difference operator

The n-order difference operator is defined as

$$\Delta^n = (1 - L)^n$$

Example:

$$\begin{aligned}\Delta^2 y_t &= (1 - 2L + L^2) y_t = (1 - L) y_t - (1 - L) L y_t \\ &= \Delta y_t - \Delta y_{t-1}\end{aligned}$$

Q: Is $\Delta^3 y_t = \Delta y_t - \Delta y_{t-2}$?