Models of Nonstationary Time Series

(Hamilton: Chapters 15)

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- Introduction
- Comparison of trend-stationary and unit root processes
- Comparison of forecast errors
- Comparison of dynamic multipliers
- Test for unit roots

Introduction

We discussed that a univarite ARMA process can be written:

$$\phi(L)y_t = c + \theta(L)\epsilon_t \qquad \epsilon_t \sim IID(0, \sigma^2)$$

- if the roots of $1 - \phi(z) = 0$ are outside the inner circle $\Rightarrow$ the process y_t is stationary and,
- it can be expressed as a MA(∞) process:

$$\begin{aligned} y_t =& \phi(L)^{-1}c + \phi^{-1}(L)\theta(L)\epsilon_t \\ =& \mu + \epsilon_t + \psi_1\epsilon_{t-1} + \psi_2\epsilon_{t-2} + \dots \\ =& \mu + \psi(L)\epsilon_t \end{aligned}$$

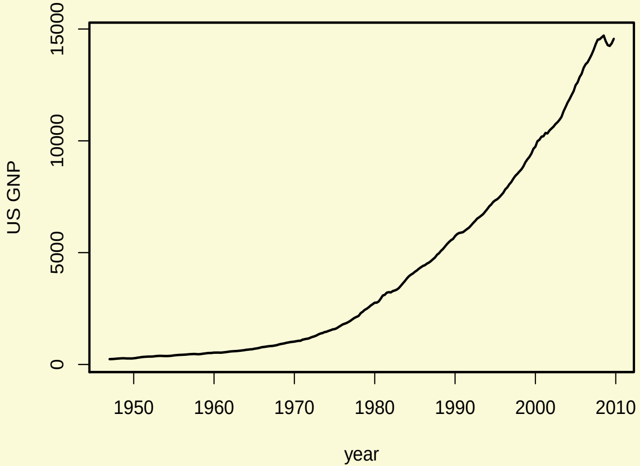
Wold representation

Introduction

$$y_t = \mu + \psi(L)\epsilon_t$$

- $\psi(L) = \sum_{j=0}^{\infty} \psi_j L^j$ where $\psi_0 = 1$
- μ is the unconditional expectation and it is a constant
- The forecast $y^*_{t+s|t}$ converges to μ when $s \to \infty$

Quarterly USA Gross National Product



Q: Do you think this series is stationary?
Do you think there is a trend? $\exp(\delta t)$, δt, $\delta t + \gamma t^2$?

Deterministic time trend

Two approaches to explain these trends:

1. A *trend stationary* process where we assume that the unconditional mean is a linear function of time $\mu_t = \alpha + \delta t$.

$$y_t = \alpha + \delta t + \psi(L)\epsilon_t$$

If we substract the trend from the model, then we have a stationary ARMA(p,q) process.

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Unit root

2. A *unit root* process ,

$$y_t = \delta + y_{t-1} + \psi(L)\epsilon_t \quad \Rightarrow \quad y_t - y_{t-1} = \delta + \psi(L)\epsilon_t$$

- the roots of $1 - \phi(z) = 0$: one on the unit circle and the rest outside.
- Then, $\phi(L) = \phi(L)^*(1 - L)$ where $\phi^*(z) = 0$ has all $p - 1$ roots outside the unit circle
- If we take first differences, then we have a stationary ARMA(p-1, q) process.
- The classical example:

$$y_t = y_{t-1} + \delta + \epsilon_t \quad \text{Random walk with drift } \delta$$

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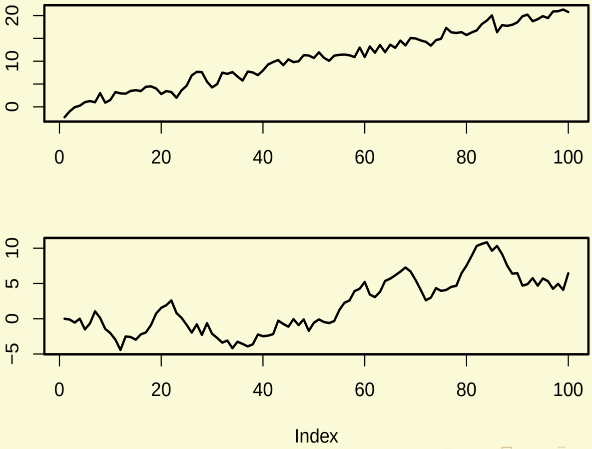
Plot trend stationary vs unit root

```
> set.seed(20)
> y = arima.sim(n=100, model=list(ar=c(0.3))) + 0.3 + 0.2*(
> #Try this command below
> #y2 = arima.sim (n = 100, model=list(ar=c(1)))
> y2<-numeric(100)
> for (i in 2:100)
+   y2[i] = y2[i-1] + rnorm(1)
```

Plot trend stationary vs unit root



Plot trend stationary vs unit root

Difference stationary

- A unit root process is a *difference stationary* process because we obtain a stationary process after first differences
- In fact, it is an ARIMA(p, 1, q)
- We also say that $y_t \sim \mathsf{I}(1)$ meaning that y_t is integrated of order 1
- For example, take the process

$$\Delta y_t = \psi(L)\epsilon_t = u_t$$

  where u_t is stationary.
- By substitution, we can write

$$y_t = y_0 + \sum_{j=1}^{T} u_j$$

  so y_t is the (integrated) sum of T stationary innovations
- We say $u_t \sim \mathsf{I}(0)$

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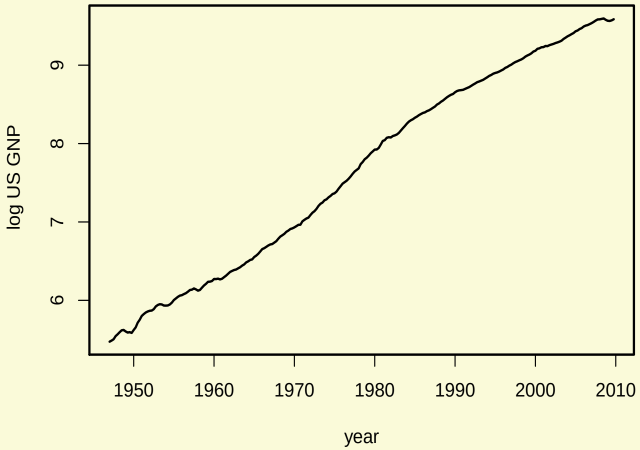
Why linear time trend processes?



Why linear time trend processes?

- From the picture, it seems that we have an exponential growth of the GNP.
- Instead of using y_t=GNP, we are going to use $\log(y_t)$, then we will have a proportional growth.
- Because we assume $y_t = e^{\delta t} \Rightarrow \log(y_t) = \delta t$
- So we will be modelling

$$\log(y_t) = \alpha + \delta t + \psi(L)\epsilon_t$$

Why unit root processes?

- We have decided to take logs of the data
- Why would we model it by:

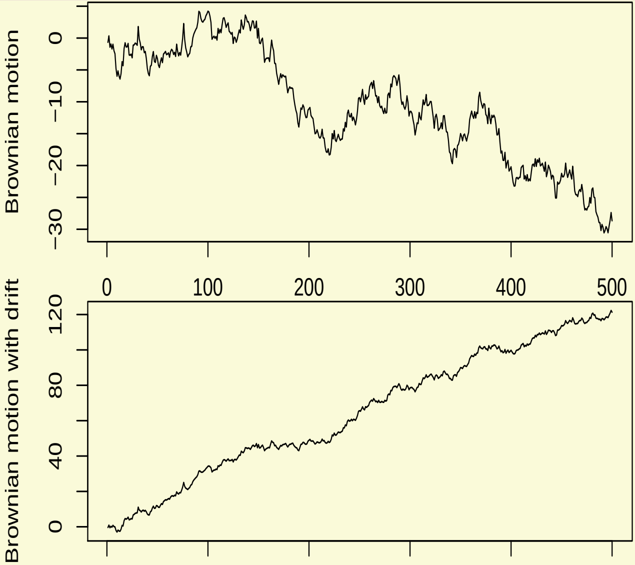
$$\log(y_t) = \delta + \log(y_{t-1}) + \psi(L)\epsilon_t$$

- Then, if the process has an unit root

$$\begin{aligned}(1-L)\log(y_t) =& \log(y_t) - \log(y_{t-1}) = \log\left(\frac{y_t}{y_{t-1}}\right) \\ =& \log\left(\frac{y_t - y_{t-1} + y_{t-1}}{y_{t-1}}\right) = \log\left(\frac{y_t - y_{t-1}}{y_{t-1}} + 1\right) \\ \approx& \frac{y_t - y_{t-1}}{y_{t-1}}\end{aligned}$$

- Then, the rate of growth of the series is a stationary stochastic process.
- In practise, we tend to multiply $\log(y_t)$ by 100. For example, $(1-L)(100\log(y_t)) = 1$, then y_t is 1% higher than y_{t-1}

Example: unit root



Comparison of trend-stationary and unit root processes

The comparison is done in terms of:

1. forecast of the series,
2. variance of the forecast error,
3. dynamic multipliers (persistence of the innovations), and
4. transformations needed to achieve stationarity.

Comparison of forecast

- The forecast of a trend-stationary process:

$$y^*_{T+s|T} = \alpha + \delta(T+s) + \psi_s \epsilon_T + \psi_{s+1}\epsilon_{T-1} + \dots$$

- As the forecast horizon s grows larger, the forecast converges in mean square to the time trend

$$E(y^*_{T+s|T} - \alpha - \delta(T+s))^2 = \sigma^2 \sum_{i=s}^{\infty} \psi_i^2 \rightarrow_{s\rightarrow\infty} 0$$

Example: if y_t is an MA(q), then $\psi_{q+1}, \psi_{q+2}, \dots = 0$. Then the forecast for $s > q \Rightarrow y^*_{T+s|T} = \alpha + \delta(T+s)$

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Comparison of forecast

- We can write y_{T+s} as

$$\begin{aligned} y_{T+s} =& (y_{T+s} - y_{T+s-1}) + (y_{T+s-1} - y_{T+s-2}) + \ldots + (y_{T+1} - y_T) + y_T \\ =& \Delta y_{T+s} + \Delta y_{T+s-1} + \ldots + \Delta y_{T+1} + y_T \end{aligned}$$

- In addition, (Δy_t is stationary)

$$\begin{aligned} \Delta y^*_{T+s|T} \equiv& E((y_{T+s} - y_{T+s-1})|\epsilon_T, \epsilon_{T-1}, \ldots) \\ =& \delta + \psi_s \epsilon_T + \psi_{s+1} \epsilon_{T-1} + \ldots \end{aligned}$$

- Putting both together

$$\begin{aligned} y^*_{T+s|T} =& (\delta + \sum_{i=s}^{\infty} \psi_i \epsilon_{T+s-i}) + (\delta + \sum_{i=s-1}^{\infty} \psi_i \epsilon_{T+s+1-i}) \\ & + \ldots + (\delta + \sum_{i=1}^{\infty} \psi_i \epsilon_{T+1-i}) + y_T \\ =& s\delta + y_T + \sum_{i=1}^{s} \psi_i \epsilon_T + \sum_{i=2}^{s+1} \psi_i \epsilon_{t-1} + \ldots \end{aligned}$$

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Examples: forecast of unit roots

For example, the random walk with drift is an unit root proccess:

$$y_t = \delta + y_{t-1} + \epsilon_t \quad \psi_0 = 1, \psi_i = 0 \;\; i > 0$$

Its forecast:

$$y^*_{T+s|T} = s\delta + y_T$$

It is expected to grow at the constant rate δ from the value at period T

Examples: forecast of unit roots

An ARMA (0,1,1)

$$y_t = \delta + y_{t-1} + \epsilon_t + \theta\epsilon_{t-1}$$

Its forecast:

$$y^*_{T+s|T} = s\delta + y_T + \theta\epsilon_T$$

It is expected to grow at the constant rate δ from the base value $y_T + \theta\epsilon_T$

The forecast an ARMA (0, 1, q)

$$y^*_{T+s|T} = s\delta + y_T + \sum_{i=1}^{min(s,q)} \theta_i\epsilon_T + \dots$$

Comparison of forecast

Conclusion:

- δ plays a similar role for the forecast of a trend stationary process and a unit root
- Basically, both forecasts converge to a linear function of the forecast horison s with slope δ
- However, for the trend stationary process the intercept is the same regardless of the value of y_T
- While the intercept of a unit root forecast depends on the last value y_T

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Comparison of forecast errors

For a **trend stationary** process:

- The forecast error

$$y_{T+s} - y^*_{T+s|T} = \epsilon_{T+s} + \psi_1 \epsilon_{T+s-1} + \ldots + \psi_{s-1} \epsilon_{T+1}$$

- The Mean Square Error (MSE) of this forecast

$$E[y_{T+s} - y^*_{T+s|T}]^2 = (1 + \psi_1^2 + \ldots + \psi_{s-1}^2)\sigma^2$$

- The MSE increases with the horizon s:

$$\lim_{s\to\infty} E[y_{T+s} - y^*_{T+s|T}]^2 = (1 + \psi_1^2 + \ldots + \psi_{s-1}^2 + \ldots)\sigma^2$$

  but it converges to a point because the process $\psi(L)\epsilon_t$ is stationary and the lim of the MSE is the unconditional variance of this process

Comparison of forecast errors

For a **unit root** process:

- The forecast error

$$y_{T+s} - y^*_{T+s|T} = \epsilon_{T+s} + (1+\psi_1)\epsilon_{T+s-1} + (1+\psi_1+\psi_2)\epsilon_{T+s-2} \ldots + (1+\sum_{i=1}^{s-1}\psi_i)\epsilon_{T+1}$$

- The Mean Square Error (MSE) of this forecast

$$E[y_{T+s} - y^*_{T+s|T}]^2 = (1 + (1+\psi_1)^2 + \ldots + (1+\sum_{i=1}^{s-1}\psi_i)^2)\sigma^2$$

- The MSE increases with the horizon s but in this case it won't converge to any fixed value but to a linear function of s

Example: forecast error of unit root

In ARIMA(0, 1,1)

$$E[y_{T+s} - y^*_{T+s|T}]^2 = \{1 + (s-1)(1+\theta)^2\}\sigma^2$$

Comparison of forecast errors

Conclusions:

- The MSE of the forecast of trend stationary process reaches a finite bound
- However, for unit roots, it grows linearly with the horizon. Therefore, the standard deviation of the forecast error grows with $\sqrt{s}$
- This means that the confident intervals of the forecast converge to a fix number in the trend stationary process but they continue growing for the unit root process

Comparison of dynamic multipliers

The persistence of innovations is different from trend stationary and unit root processes.
Q: What is the effect on y_{t+s} if ϵ_t were to increase by one unit and the rest of the ϵ unaffected?

$$\frac{\partial y_{t+s}}{\partial \epsilon_t}$$

- For **trend stationary** proceses, the effect wears off:

$$\lim_{s \to \infty} \frac{\partial y_{t+s}}{\partial \epsilon_t} = \lim_{s \to \infty} \psi_s = 0$$

- For **unit root** processes, the effect is permanent

$$\lim_{s \to \infty} \frac{\partial y_{t+s}}{\partial \epsilon_t} = \lim_{s \to \infty} (1 + \psi_1 + \ldots + \psi_{s-1} + \psi_s) = \psi(1)$$

Example of unit root dynamic multiplier

Use the $y_t = 100\times$ log of US GNP and estimate an ARIMA(4,1,0)

```
> gnp.unitroot<-arima(100*gnpln, order=c(4,1,0))
> psi= round(coefficients(gnp.unitroot),3)
> psi
```

```
  ar1    ar2    ar3    ar4
0.585  0.274 -0.083  0.117
```

$$\psi(1) = \frac{1}{\phi(1)} = \frac{1}{1 - 0.585 - 0.274 - 0.083 - 0.117} = 9.346$$

The permanent effect of a one unit change in ϵ_t on the level of GNP is estimated to be around 9%

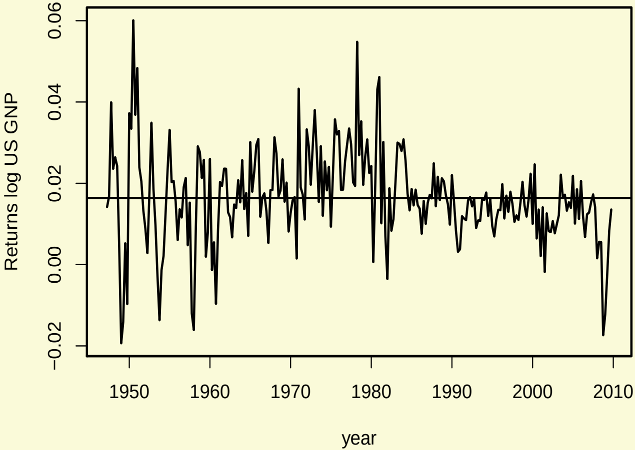
Transformations to achieve stationarity

The transformation needed to achieve stationarity is also different in trend stationary and unit root processes.

- For a **trend stationary process**, we have to substract the trend. The remaining process is stationary
- For a **unit root process**, we need to take first differences

Transformations to achieve stationarity

Plot the first difference of the log US GNP



Where does the nonstationarity come from?

- Does nonstationarity comes from a time trend or a unit root?
- For example, we know that the US GNP is nonstationary:
  - If it comes from a unit root, the economic recessions will have permanent consequents for the level of future GNP
  - If it comes from a time trend, the effects will be temporary downturns with the lost output eventually made up during the recovery
- Some authors have argue that answering whether a nonstationary process has a unit root cannot be answered on the basis of a finite sample. Q: why is this?

Where does the nonstationarity come from?

Let say we have the true model with a unit root:

$$y_t = y_{t-1} + \epsilon_t \tag{1}$$

There is a stationary model (false model):

$$y_t = \phi y_{t-1} + \epsilon \quad |\phi| < 1 \text{ but very close to 1} \tag{2}$$

Q: How do we differenciate between these two processes from our data set?

Where does the nonstationarity come from?

The s-period-head forecast of the unit root process (1)

$$y^*_{T+s|T} = y_T \qquad MSE(s) = E(y^*_{T+s|T} - y_T)^2 = s\sigma^2$$

The corresponding forecast of the stationary process (2)

$$y^*_{T+s|T} = \phi^s y_T \qquad MSE(s) = (1 + \phi^2 + \phi^4 + \ldots + \phi^{2(s-1)})\sigma^2$$

If ϕ is close to 1, all formula are similar $\Rightarrow$ It is impossible to differentiate one forecast from the other.

Where does the nonstationarity come from?

- For any unit root process and a given T, there exists a stationary process that will be impossible to distinguish from the nonstationary process
- Conversely, for every stationary process and a given T, there exists a unit root process that will be impossible to distinguish from the stationary process.

However, we can ask: Does innovations have a significant effect on the level of the series over a specified finite horizon?

Where does the nonstationarity come from?

- For a fixed horizon, for example $s = 3$, there exists a sample size T (half a century observations from WWII) such that we can meaningfully inquire whether $\partial y_{T+s}/\partial \epsilon_T$ is close to zero.
- We do not know whether the data was generated by (1) or (2) but we can measure the persistence of the series
- For example, we can assume that the proces follows an AR(1) process and test the hypothesis $H_0 : \phi = 1$
- Of course the test would have a low power to distinguish between $\phi = 0.99999$ and $\phi = 1$
- We can test H_0 : Is $\{y_t\}$ a AR(1) process with an unit root? but we cannot test H_0 : Is $\{y_t\}$ a unit root process?

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Where does the nonstationarity come from?

Q: Are there any other sources of nonstationarity?

A: YES

- Fractionally integrated processes
- Processes with occasional, discrete shifts in the time trend

Fractional integration

An integrated process of order d: I(d) is represented by

$$(1-L)^d y_t = \psi(L)\epsilon_t$$

with $\sum |\psi_j| < \infty$.

- We usually assume that $d = 1$ (unit root) or at the most $d = 2$.
- But can $0 < d < 1$ be any rational number? For example $d = 0.3$ or $d = 0.7$? What does it mean?
- It means that the process has long memory.
- If $d < 1/2 \Rightarrow$ stationary process with long memory
- If $d \geq 1/2 \Rightarrow$ nonstationary process with long memory
- These could be estimated with large-order ARMA processes. Insted, we take fractional differences first to use smaller order (law of parsimony)

Fractional integration

If a process is I(0.7):

$$(1-L)^{0.7} y_t = \psi(L)\epsilon_t$$

then 0.7= 1-0.3.

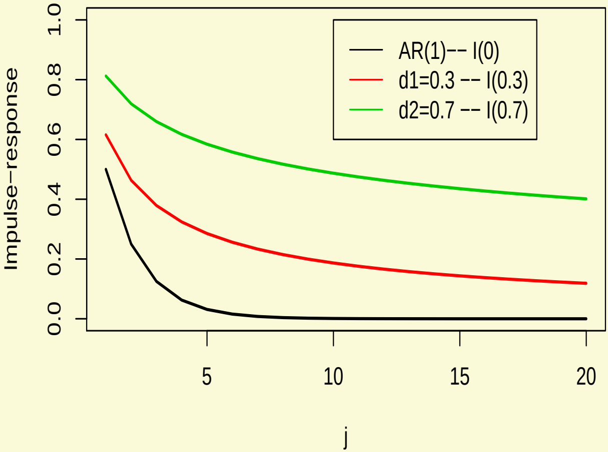
So we can get a new process as the first difference of y_t:
$r_t = (1-L)y_t$ and

$$(1-L)^{-0.3} r_t = \psi(L)\epsilon_t$$

where $d = -0.3 < 1/2$.

Long memory can arise from aggregation of other processes (Granger, 1980). That could be the reason why it is found in the absolute returns of indexes such as the S&P 500.

Fractional integration



Occasional breaks in trends

- The unit root means that events with permanent effect on y_t are occurring all the time.
- Perron (1989) and Rappoport and Reichlin (1989) argue that it makes more sense to believe that this permanent effect events only occur rarely
- They propose:

$$y_t = \begin{cases} \alpha_1 + \delta t + \epsilon_t & \text{for } t < T_0 \\ \alpha_2 + \delta t + \epsilon_t & \text{for } t \geq T_0 \end{cases} \tag{3}$$

- This series would appear to exhibit unit root nonstationarity on the basis of an unit root test.

Occasional breaks in trends

Anothe way of writting (3):

$$\Delta y_t = \xi_t + \delta + \epsilon_t - \epsilon_{t-1} \qquad \xi_t = \begin{cases} 0 & t \neq T_0 \\ \alpha_2 - \alpha_1 & t = T_0 \end{cases} \tag{4}$$

If we view ξ_t as a random variable

$$\xi_t = \begin{cases} 0 & \text{with probability } 1 - p \\ \alpha_2 - \alpha_1 & \text{with probability } p \end{cases}$$

where p is quite small. Then, (4) can be written as:

Occasional breaks in trends

- Lam (1990) assume that the US real GNP had a trend which slope was modelled with a Markov chain.

$$y_t = n_t + z_t$$

$$n_t = n_{t-1} + \alpha_0 + \alpha_1 S_t \qquad S_t \text{ is 0 or 1 with prob } P$$

$$P = \begin{pmatrix} p & 1-p \\ q & 1-q \end{pmatrix} \qquad \text{where } P_{ij} = P(S_t|S_{t-1})$$

$$z_t = \phi_1 z_{t-1} + \phi_2 z_{t-2} + \phi_3 z_{t-3} + \epsilon_t$$

- According to his study, events that permanently changed the level of GNP coincided with the recessions of 1957, 1973 and 1980.