PROBLEM SET 6

Problem 1 (Pooled OLS, Random Effects and Fixed Effects)

(a) For $\gamma = 0$ we have $\mathbb{E}(x_{it}u_{i\tau}) = 0$. Thus, we find for $n \to \infty$

$$\hat{\beta}_{\text{OLS}} - \beta = \frac{\frac{1}{n}\sum_{i=1}^n x_i' u_i}{\frac{1}{n}\sum_{i=1}^n x_i' x_i} \to_p \frac{\mathbb{E}x_i' u_i}{\mathbb{E}x_i' x_i} = 0,$$

i.e. $\hat{\beta}_{\text{OLS}}$ is consistent. Analogously, one finds that $\hat{\beta}_{\text{GLS}}$ and $\hat{\beta}_{\text{WG}}$ are consistent, i.e. all three estimators are consistent.

(b) The GLS estimator has the smallest asymptotic variance.

(c) For $\gamma \neq 0$ we have $\mathbb{E}(x_{it}u_{i\tau}) \neq 0$, and in particular $\mathbb{E}(x_i' u_i) \neq 0$ and $\mathbb{E}(x_i' \Sigma^{-1} u_i) \neq 0$. Thus, the OLS and GLS estimator are NOT consistent in that case.

The WG estimator is still consistent, because $x_i' M u_i = x_i' M \varepsilon_i$, i.e. M projects out α_i completely, and therefore $\mathbb{E}(x_i' M u_i) = \mathbb{E}(x_i' M \varepsilon_i) = 0$. We thus have as $n \to \infty$

$$\hat{\beta}_{\text{WG}} - \beta = \frac{\frac{1}{n}\sum_{i=1}^n x_i' M u_i}{\frac{1}{n}\sum_{i=1}^n x_i' M x_i} = \frac{\frac{1}{n}\sum_{i=1}^n x_i' M \varepsilon_i}{\frac{1}{n}\sum_{i=1}^n x_i' M x_i} \to_p \frac{\mathbb{E}x_i' M \varepsilon_i}{\mathbb{E}x_i' M x_i} = 0.$$

Thus, only $\hat{\beta}_{\text{WG}}$ is consistent.

(d) We have $\mathbb{E}(x_{it}\tilde{u}_{it}) = 0$ and $\mathbb{E}(w_i \tilde{u}_{it}) = 0$, i.e. the regressors are exogenous, and one can also show that they are non-collinear. Thus, the pooled OLS estimator proposed here is indeed a consistent estimator for β and γ.

Problem 2 (Pooled OLS and Random Effects)

(a) As $n \to \infty$ we have

$$\hat{\beta}^{\mathrm{GLS}} - \beta = \frac{\frac{1}{n}\sum_{i=1}^{n} x_i' \Sigma^{-1} u_i}{\frac{1}{n}\sum_{i=1}^{n} x_i' \Sigma^{-1} x_i} \to_p \frac{\mathbb{E}(x_i' \Sigma^{-1} u_i)}{\mathbb{E}(x_i' \Sigma^{-1} x_i)} = 0,$$

where we applied the WLLN and the fact that $\mathbb{E}(x_{it} u_{i\tau}) = 0$. We have thus shown $\hat{\beta}^{\mathrm{GLS}} \to_p \beta$ as $n \to \infty$, i.e. consistency of $\hat{\beta}^{\mathrm{GLS}}$. The consistency argument for $\hat{\beta}^{\mathrm{OLS}}$ is analogous.

(b) $$\Sigma = \mathbb{E}(u_i u_i' | x_i) = \begin{pmatrix} \sigma_\varepsilon^2 + \sigma_\alpha^2 & \sigma_\alpha^2 \\ \sigma_\alpha^2 & \sigma_\varepsilon^2 + \sigma_\alpha^2 \end{pmatrix},$$

$$\Sigma^{-1} = \frac{1}{2\sigma_\varepsilon^2 \sigma_\alpha^2 + \sigma_\varepsilon^4} \begin{pmatrix} \sigma_\varepsilon^2 + \sigma_\alpha^2 & -\sigma_\alpha^2 \\ -\sigma_\alpha^2 & \sigma_\varepsilon^2 + \sigma_\alpha^2 \end{pmatrix}.$$

(c) As $n \to \infty$ we apply the WLLN and the CLT to obtain

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} x_i' \Sigma^{-1} x_i &\to_p \mathbb{E}(x_i' \Sigma^{-1} x_i), \\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i' \Sigma^{-1} u_i &\Rightarrow \mathcal{N}\left[0, \mathbb{E}(x_i' \Sigma^{-1} u_i u_i' \Sigma^{-1} x_i)\right] \\ &\quad \sim_d \mathcal{N}\left[0, \mathbb{E}(x_i' \Sigma^{-1} x_i)\right]. \end{aligned}$$

Applying Slutsky's theorem then gives

$$\sqrt{n}\left(\hat{\beta}^{\mathrm{GLS}} - \beta\right) = \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i' \Sigma^{-1} u_i}{\frac{1}{n}\sum_{i=1}^{n} x_i' \Sigma^{-1} x_i} \Rightarrow \mathcal{N}\left\{0, \left[\mathbb{E}(x_i' \Sigma^{-1} x_i)\right]^{-1}\right\}.$$

By analogous reasoning one finds that

$$\sqrt{n}\left(\hat{\beta}^{\mathrm{OLS}} - \beta\right) \Rightarrow \mathcal{N}\left\{0, \mathbb{E}(x_i' x_i)^{-1} \mathbb{E}(x_i' \Sigma x_i) \mathbb{E}(x_i' x_i)^{-1}\right\}.$$

Using the distributional assumptions on x_{it}, ε_{it} and α_i we can express the asymptotic variances of the estimators as follows

$$
\begin{aligned}
\text{AsyVar}(\hat{\beta}^{\text{OLS}}) &= \mathbb{E}(x_i'x_i)^{-1}\mathbb{E}(x_i'\Sigma x_i)\mathbb{E}(x_i'x_i)^{-1} \\
&= \left(2\sigma_x^2\right)^{-1}\left(2\sigma_x^2(\sigma_\varepsilon^2+\sigma_\alpha^2)\right)\left(2\sigma_x^2\right)^{-1} \\
&= \frac{\sigma_\varepsilon^2+\sigma_\alpha^2}{2\sigma_x^2}. \\
\text{AsyVar}(\hat{\beta}^{\text{GLS}}) &= \left[\mathbb{E}(x_i'\Sigma^{-1}x_i)\right]^{-1} \\
&= \left[\frac{2\sigma_x^2(\sigma_\varepsilon^2+\sigma_\alpha^2)}{2\sigma_\varepsilon^2\sigma_\alpha^2+\sigma_\varepsilon^4}\right]^{-1} \\
&= \frac{2\sigma_\varepsilon^2\sigma_\alpha^2+\sigma_\varepsilon^4}{2\sigma_x^2(\sigma_\varepsilon^2+\sigma_\alpha^2)}.
\end{aligned}
$$

We have

$$
\text{AsyVar}(\hat{\beta}^{\text{OLS}}) - \text{AsyVar}(\hat{\beta}^{\text{GLS}}) = \frac{\sigma_\alpha^4}{2\sigma_x^2(\sigma_\varepsilon^2+\sigma_\alpha^2)} > 0,
$$

Thus, $\hat{\beta}^{\text{GLS}}$ has the smallest asymptotic variance.