PROBLEM SET 3 SOLUTIONS

Problem 1 (The Probit Model)

(a)

$$P(Y=1|X=x)=P(u>-x\beta|X=x)=P(u<x\beta|X=x)=\Phi(x\beta),$$

where the second equality sign holds by the symmetry of a normal distribution.

$$P(Y=0|X=x)=1-\Phi(x\beta).$$

(b)

$$\frac{\partial P(Y=1|X=x)}{\partial x_k}=\beta_k\phi(x\beta)$$

(c)

$$\begin{aligned}l(\beta)&=\sum_{i=1}^{n}y_i\log\ P(Y=1|X=x)+(1-y_i)\log\ P(Y=0|X=x)\\&=\sum_{i=1}^{n}y_i\log\Phi(x_i\beta)+(1-y_i)\log[1-\Phi(x_i\beta)]\end{aligned}$$

(d)

(1) The OLS provides partial effects that are constant at all values of x.
(2) The OLS leads to predicted probabilities that can lie outside [0,1] for some values of x.

(e)

We require that $E[X'X]$ is invertible. To see why, note that from

$$P(Y=1|X=x)=\Phi(x\beta)$$

we have

$$\beta=E[X'X]^{-1}X'\Phi^{-1}(P(Y=1|X))$$

and $P(Y=1|X)$ is identified under random sampling. Alternatively, it can be shown that the expected log-likelihood has a unique maximizer if and only if $E[X'X]^{-1}$ exists.

Problem 2 (MLE and the Information Equality)

(a) We have

$$\begin{aligned}\log f(y|\theta) &= \log \prod_{i=1}^{n} f(y_i|\theta) = \sum_{i=1}^{n} \log f(y_i|\theta) \\ &= \sum_{i=1}^{n} \left[-\log \sqrt{2\pi} - \frac{1}{2}(y_i - \log\theta)^2 \right] \\ &= -n\log\sqrt{2\pi} - \frac{1}{2}\sum_{i=1}^{n}(y_i - \log\theta)^2.\end{aligned}$$

The FOC reads $\frac{1}{\hat{\theta}} \sum_{i=1}^{n}(y_i - \log\hat{\theta}) = 0$. Solving this gives $\hat{\theta} = \exp(\frac{1}{n}\sum_{i=1}^{n} y_i)$.

(b) We calculate

$$H(y_i, \theta) = \frac{\partial^2 \log f(y_i|\theta)}{\partial\theta^2} = -\frac{1}{\theta^2}(y_i - \log\theta) - \frac{1}{\theta^2}.$$

(c) We have $\mathbb{E}y_i = \log\theta_0$. Therefore $\mathbb{E}H(y_i, \theta_0) = -\frac{1}{\theta_0^2}$.

(d) $\text{AsyVar}(\sqrt{n}\hat{\theta}) = -\mathbb{E}H(y_i, \theta_0)^{-1} = \theta_0^2$.

(e) Using that $\mu(\theta) = \log\theta$ and the delta method we find

$$\text{AsyVar}(\sqrt{n}\hat{\mu}) = \left[\frac{d}{d\theta}\bigg|_{\theta=\theta_0} \log\theta \right]^2 \text{AsyVar}(\sqrt{n}\hat{\theta}) = \frac{1}{\theta_0^2}\text{AsyVar}(\sqrt{n}\hat{\theta}) = 1.$$